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The Stepped Horn

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THE STEPPED HORN

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ABSTRACT

Of all devices designed to mechanically amplify ultrasonic vibrations the stepped horn is apparently the most efficacious. In addition it has many other interesting properties. It is the only resonant device known fin which the resonant frequency can be changed by a factor of two to one without changing the over-all dimensions. Furthermore, the numerical relationship between overtones may easily be changed. The displacement magnification may also be varied widely. For certain horn materials Q's of 25,000 have been achieved.

The results of a mathematical analysis are presented together with experimental verification.

I. **INTRODUCTION**

The work reported on here is the result of a search for a means of obtaining comparatively large amplitudes of motion in a solid at ultrasonic frequencies. The device that resulted turned out to have interesting and unusual properties and led to the solutions to equations having several applications in the field of ultrasonics.

Il. **THE STEPPED HORN**

Existing magnetostrictive and piezoelectric transducers produce motion having too small an amplitude for applications such as ultrasonic machining ing. welding. and liquid atomizing. In such applications It is common practice to amplify the mechanical motion by means of a tapered resonant horn, and the shape of the taper most commonly chosen is exponential. Other configurations have been used but It would seem that a very useful one, the stepped horn, has been overlooked. In the literature it appears that only Balamuth has even mentioned it, and Its full potential has apparently not been realized.

The stepped horn, in Its simplest form, consists of two cylinders of different diameter placed end to end concentrically. In practice the whole in usually machined out of a solid. Perhaps Its most useful characteristic Is the large motion amplification that is obtainable. Fig. 1 illustrates. and compares briefly the magnification ofthe exponential, straight taper, and stepped horns. It will be seen that, for a given ratio of end diameters, the stepped horn offers the greatest displacement magnification. Or, conversely, for a given desired magnification and large end diameter, the stepped horn gives a larger tip diameter than other horns. In many applications this large tip diameter is very desirable, but even apart from this, the facts that a larger dimension may be used and that the geometrical configuration is simple mean that the stepped horn lends itself very well to both experimental and production work. A machinist can produce-one in minutes without templates or special attachments.

III. **DERIVATION OF RESONANT FREQUENCIES**

It is of interest to know, for any configuration, what the resonant frequencies and magnification. One may deduce qualitatively how the frequency will vary with position of the step.

Consider a uniform bar as shown in Fig. 2(a). This will have a certain frequency in longitudinal resonance. If a portion of the bar is removed as in Fig. 2(b)t the frequency will be increased. At the point where each section is a quarter wave the frequency will return to the original frequency. Reversing the process so as to start with a slender rod as in Fig. 2(d), adding mass will reduce the frequency as in Fig. 2(e). Thus the frequency curve of Fig. 2(f) may be deduced.

In order to obtain a quantitative analysis, resort is made to the wave equation. In an appendix to this paper It is shown that the resonant frequencies may be found from the frequency equation

$$R\sin 2\pi qp\cos 2\pi q(l-p) + \cos 2\pi qp\sin 2\pi q(l-p) = 0$$

where

$R = S_a/S_b$ $p = a/1$ $q = fl/v$,

and S_a and S_b respectively are the areas of the larger and smaller cross-sections, a Is the length of the section having the larger area, l is the total length, v the sound velocity, and f the frequency.

This equation may be solved by using Fig. 3 where q is plotted as a function of p for several values of R. The several values of q f or each value of p and R correspond to the overtones.

It will be observed that the resonant frequency can be varied substantially (by a factor of 2:1 In the case of the fundamental) without changing the overall dimensions of the horn. In addition, the ratio of overtone to fundamental frequency may be varied at will over a wide range, suggesting the use of such horns in filters having wore than one passband. The thought has also been advanced that use might be made of the variable overtone property to Improve the timbre of electronic musical percussion Instruments. Another possible use of this property would be to produce overtones that am exact multiples of the fundamental by introducing a small area change into an otherwise uniform bar - It will be recalled that, even in uniform bars. because of the difficulty of setting up pure modes of motion, the overtones are not exact multiples of the fundamental.

IV. **DISPLACEMENT MAGNIFICATION**

In the appendix it is shown that the displacement magnification, defined an the magnitude of the ratio of the displacement at the small end to the displacement at the large end is given by

$$M = \left| \frac{\cos 2\pi qp}{\cos 2\pi rq(l-p)} \right|$$

where the values of q for a specific case are obtained from the frequency equation. In Fig. 4 is given the solution to this equation as a function of p for several values of R and for several overtones. The maximum possible magnification under any conditions is R, the ratio of the areas, and the smallest magnification is 1. The magnification varies as the position of the step is changed, but the maximum value R occurs n times, where n is the order of the overtone.

V. **SOLUTION OF ASSOCIATED PROBLEMS**

In passing it may be noted that the solutions shown in Fig. 3 and 4 may be used to solve other problems. For the general case where the two sections of the bar are of different materials as well as different areas, solutions for resonant frequency and magnification may be obtained provided that the following definitions are used:

$$R = \frac{S_a \rho_a v_a}{S_b \rho_b v_b}$$

$$p = \frac{a}{a + (v_a/v_b)(l-a)}$$

$$q = f\,[\,a/v_a + (l-a)/v_b]$$

where the subscripts a and b refer to the area, density. and velocity in sections of length a and b = l - a. It is then possible, for instance, to compute frequency changes due to a driving transducer or to find the exact solution to problems Involving mass loading.

VI. **COMPARISON OF THEORETICAL AND EXPERIMENTALRESULTS**

In order to cheek the validity of the solutions here given, a series of stepped horns was made with the step in different positions. These were driven by piezoelectric drivers, and the resonant frequencies. magnifications, and Q's measured. Fig. 5 shows the experimental frequency points against the theoretical curves. The general agreement is very good, but two areas of disagreement will be noted. The first is at higher frequencies where the diameter of the horn is an appreciable fraction of a wavelength and transverse modes are starting to establish themselves. The second is at small values of p=a/l. The disagreement here is due to the relatively much larger effect of the driving transducer. which was neglected In the calculations, In this region. For small values of p the measured frequency decreased sharply. However, a computation taking the transducer into account at p = 0 showed that indeed the f frequency should decrease at this point. The computed point is marked by an X in Fig. 5.

VII. **Q VALUES**

A number of Q measurements were made on slumInum stepped horns. The value of Q usually given (Ref.2) for aluminium Is about 10,000, yet Q's in excess of 25,000 were found for these horns. The mechanism of dissipation is not well understood, but it is probably correct to say that q depends not only upon the material, but also upon the geometrical configuration and mode of vibration. Q is defined as the ratio or energy stored to energy dissipated per cycle. Stored energy certainly depends upon configuration and mode through the equivalent mass and compliance parameters. It, seems unlikely that dissipated energy would vary in such a way an to maintain Q constant when mode and configuration are changed. Thus. the high Q's noted are probably not in contradiction with those found In the literature. It was observed that Q varied appreciably with level or drive. even at the low levels where most measurements were made. This was probably due to the substantially higher stresses existing in the narrow portion of the horns.

VIII. **THE STRESS PROBLEM**

One disadvantage inherent in all horn type motion amplifiers is the high stress set up when large magnifications are employed. In the case of the stepped horn the maximum magnification, M_{max} = R, is obtained when the step is placed at a position given by a/l = (2m-1)2n where n is the overtone order, and m =1, 2, ——n. These are also the nodal points. and, unfortunately, the points of greatest stress. If high amplitudeof motion are required, it is usually necessary to take some loss In magnification and move the high stress to a larger cross-section. or to reinforce the stress area by means of fillets or flanges. Because this in done in a nodal area, the effect of the additional mass will be negligible provided it is concentrated in the nodal region.

IX. CONCLUSIONS

A device has been described which is capable of giving large magnification of mechanical motion at its resonant frequencies. It has been shown that substantial variation of the resonant frequency can be obtained without changing the over-all dimensions, and that the relationship between overtone frequencies can be radically altered. Solutions to the frequency and magnification equations, which may also be applied to other problems in acoustics, have been obtained in graphical form.

X . APPENDIX

Derivation or the Frequency Equation for Composite Bar.

Consider the composite bar shown in Fig. 7. The shape of the cross-section in arbitrary, but is assumed to be the same for all sections. Each section has a different density, elastic modulus, cross-section area, and length, as shown.

If plane cross-sections stay plane, and particle motion Is permitted only parallel to the x-axis, the differential equation for plane-wave , motion may be written

$$\frac{\partial^2\eta}{\partial x^2} = \frac{r\,\partial^2\eta}{Y\partial t^2}$$

where η is the particle displacement In the +x direction. If a sinusoldal time variation is assumed this reduces to
$\eta'' + k^2\eta = 0$ which has the solution
$\eta = A\cos kx + B\sin kx$
This equation is applied in turn to each section or the bar. The boundary conditions are that the strains shall be vanishing at $x = -(a+b)$ and at $x = c$, and that at $x = -b$ and $x = 0$, the force and displacement shall be continuous. This leads to a set of six simultaneous equations whose determinant is

$$\begin{matrix} \sin k_a(a+b) & \cos k_a(a+b) & 0 & 0 & 0 & 0 \\ \cos k_a b & -\sin k_a b & -\cos k_b b & \sin k_b b & 0 & 0 \\ \alpha\sin k_a b & \alpha\cos k_a b & -ß\sin k_b b & -ß\cos k_b b & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & ß & 0 & -y \\ 0 & 0 & 0 & 0 & -\sin k_c c & \cos k_c c \end{matrix}$$

where $a = S_a\rho_a v_a$ $ß = S_b\rho_b v_b$ $y = S_c\rho_c v_c$

For a nontrivial solution, this determinant must be zero.
This condition leads to the frequency equation

$$ß^2\cos k_a a \sin k_b b \cos k_c c + \alphaß\sin k_a a \cos k_b b \cos k_c c + ßy\cos k_a a \cos k_b b \sin k_c c + \alpha y\sin k_a a \sin k_b b \sin k_c c = 0$$

Special Cases

1. The stepped horn

By allowing c to go to zero in Equation (1) and assuming sections a and b to be of the same material, the frequency equation becomes

$$S_a\sin k_a \cos k_b + Sb\cos k_a \sin k_b = 0$$

This may be rewritten as

$$R \sin 2\pi qp \cos 2\pi q(l-p) + \cos 2\pi qp \sin 2\pi q(l-p) = 0$$

where
$R = \frac{S_a}{S_b}$ $q = \frac{fl}{v}$ $p = \frac{a}{l}$ $l = a + b$

Fig. 3 shows the first several values of q as a function of p for several values of R.

For the stepped horn, in addition to knowing the frequencies of resonance, it is desirable to know the displacement magnification, M. Displacement magnification in defined as the magnitude of the ratio of the displacements at opposite ends of the horn. Thus

$$M = \left|\frac{\eta_{b\,|\,x=0}}{\eta_{a\,|\,x=-(a+b)}}\right| = \left|\frac{\cos ka}{\cos kb}\right|$$

$$= \left|\frac{\cos 2\pi qp}{\cos 2\pi rq(l-p)}\right|$$

where the value of q to be used is obtained from the solution of the frequency equation. However, M may be found directly from Fig. 4 an a function of p, R, and order of overtone.

2. Other Cases
If the two portions of the composite bar (for the case c=0) are of different materials, the solutions given in Fig. 3 and Fig. 4 may still be used provided that the definitions of R, p and q given in equation (2) are modified

$$R = \frac{S_a\rho_a v_a}{S_b\rho_b v_b}$$

$$p = \frac{a}{a + (v_a/v_b)(l-a)}$$

$$q = f\,[\,a/v_a + (l-a)/v_b]$$

Thus, Fig. 3 and 4 may be used to compute frequency and magnification for any two-section composite bar. It is interesting to note that magnification may be obtained with no step if the materials are different.

These charts may also be used to compute the exact solution to mass-loading problems rather than use the approximation based on the assumption that standing waves are not set up in the mass.

Fig. 3 and 4 cannot be used as a solution in the case of the three-section bar except in one or two very specialized cases. The three-section equation was set up to study a stepped horn with a transducer attached.

X1. **ACKNOWLEDGMENTS**

The author wishes to thank Mr. Jim Eaglish, Mr. Ivan Gruet and Mr. Otto Schricker for their cooperation in the computation of the data for resonant frequency and magnification.

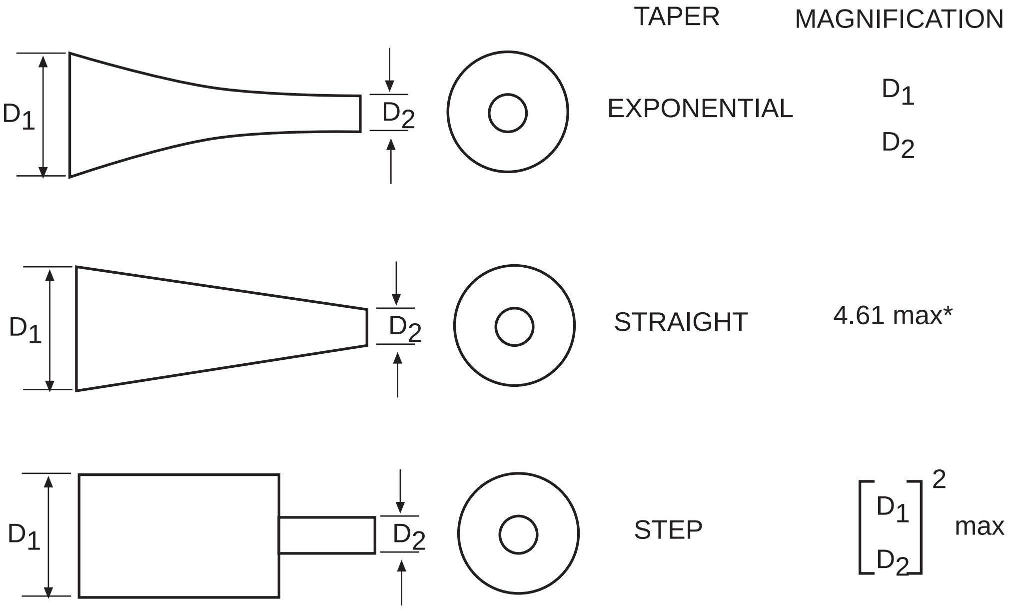


Fig. 1 Comparison of displacement magnification of exponential, straight and stepped horns

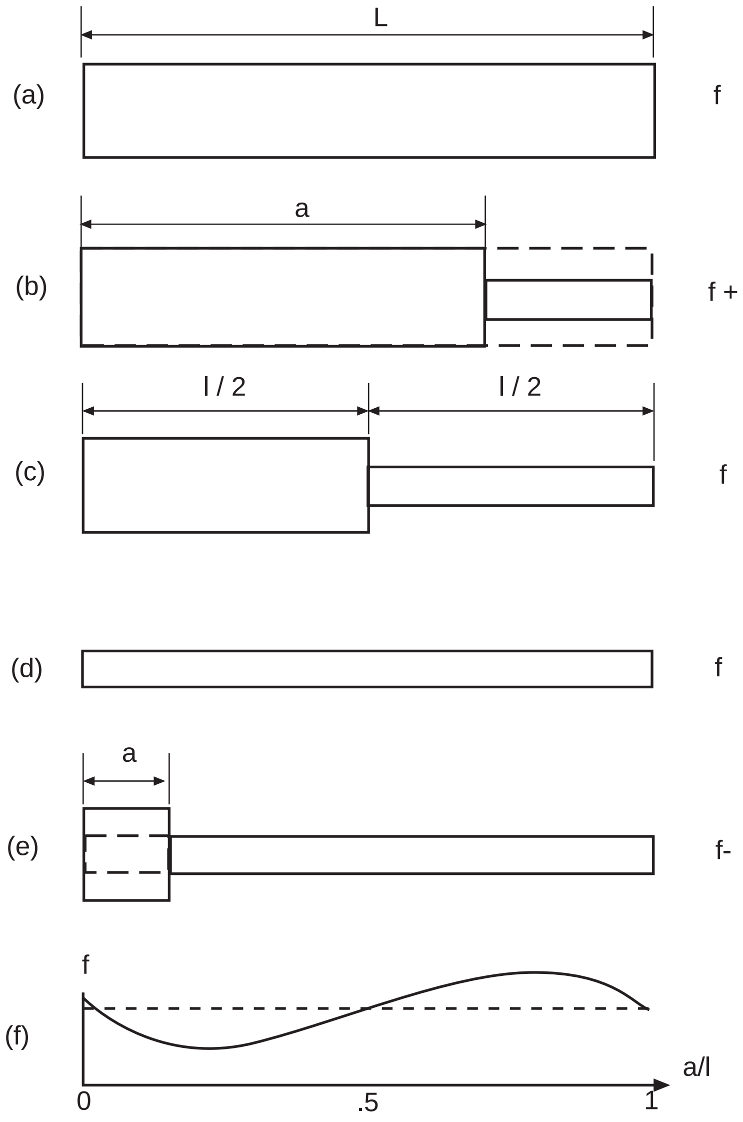


Fig. 2 Qualititaive deduction of resonant frequencies of stepped horn

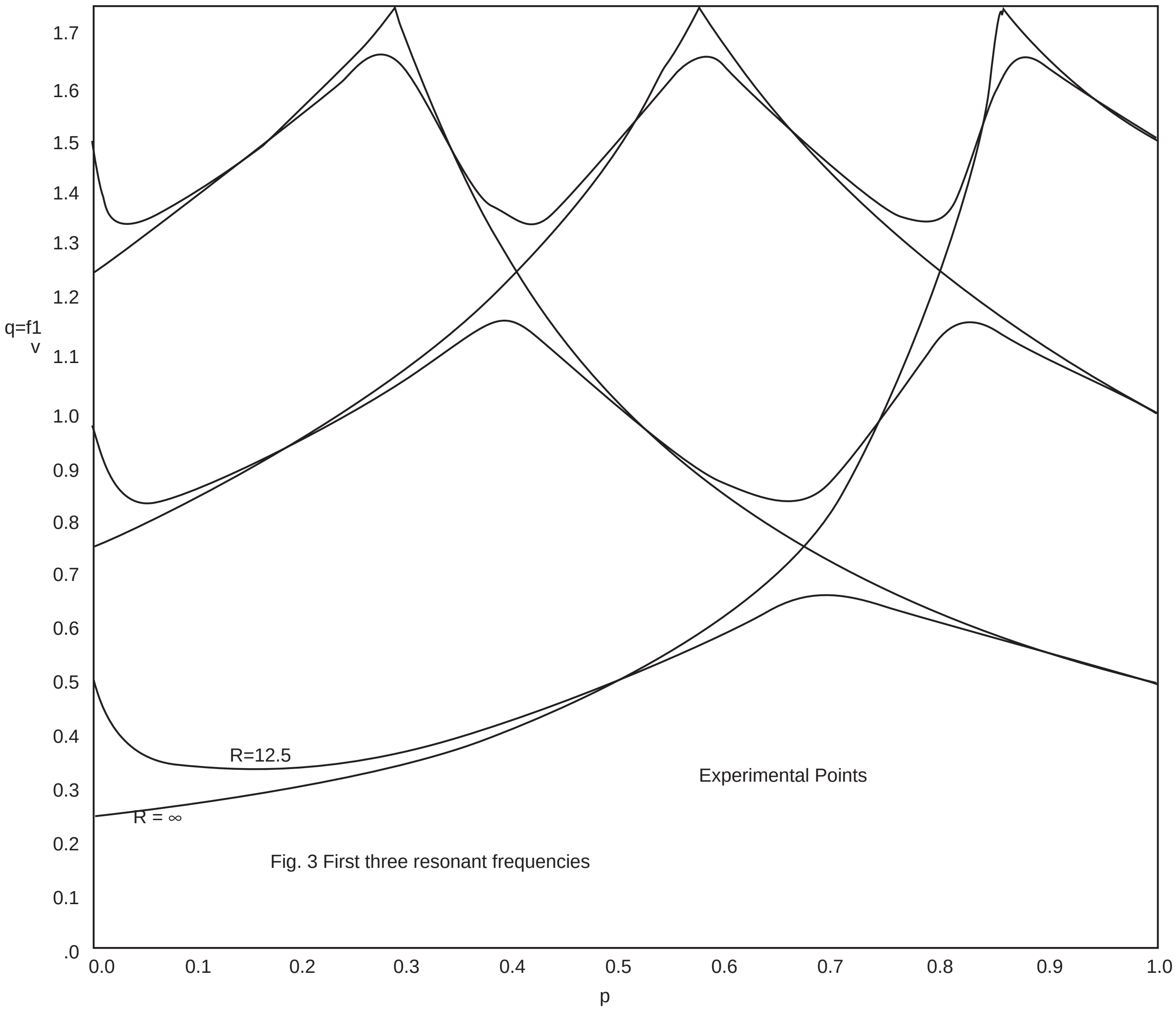


Fig. 3 First three resonant frequencies

Fig. 4 Sample Magnification R=12.5

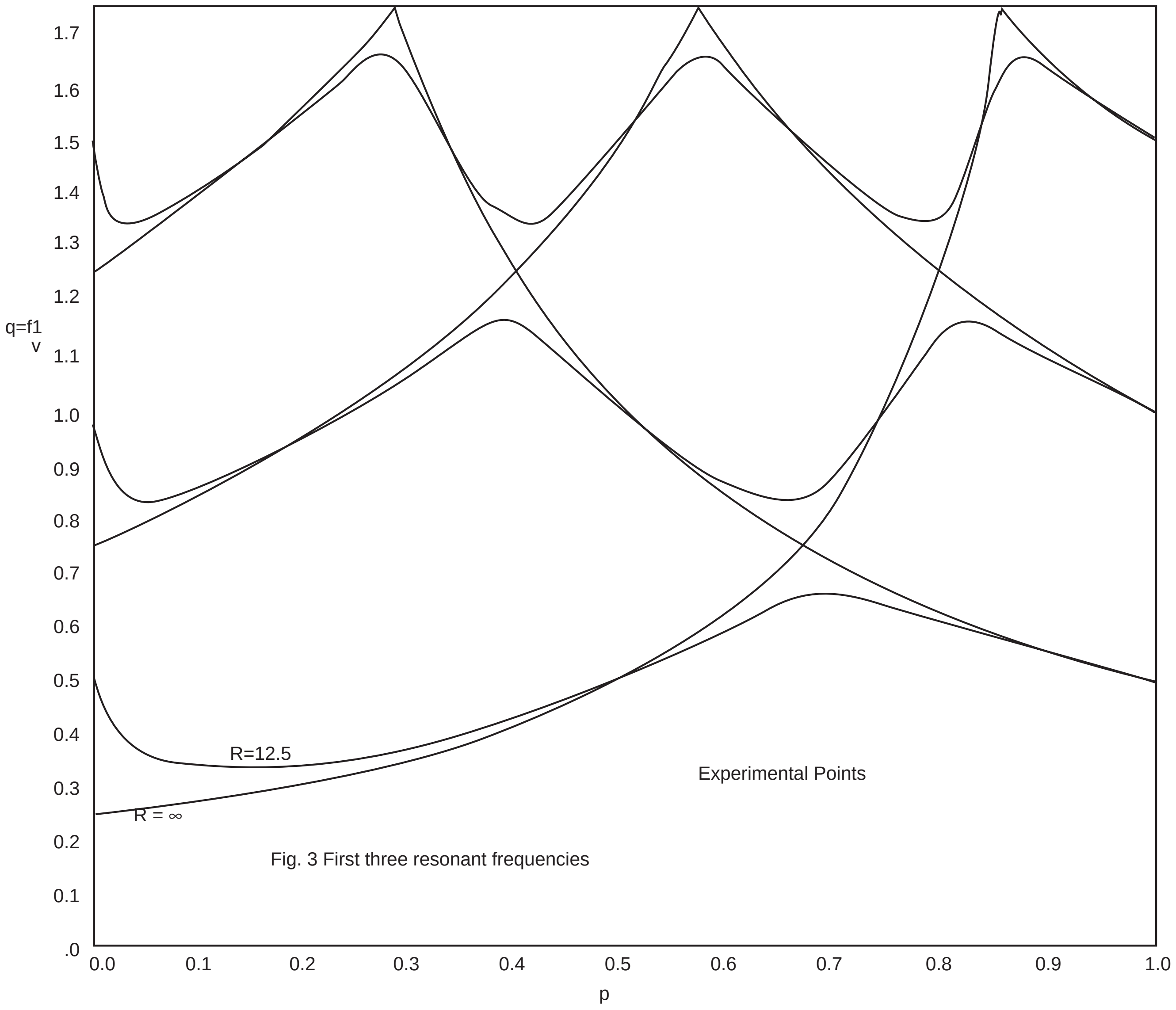
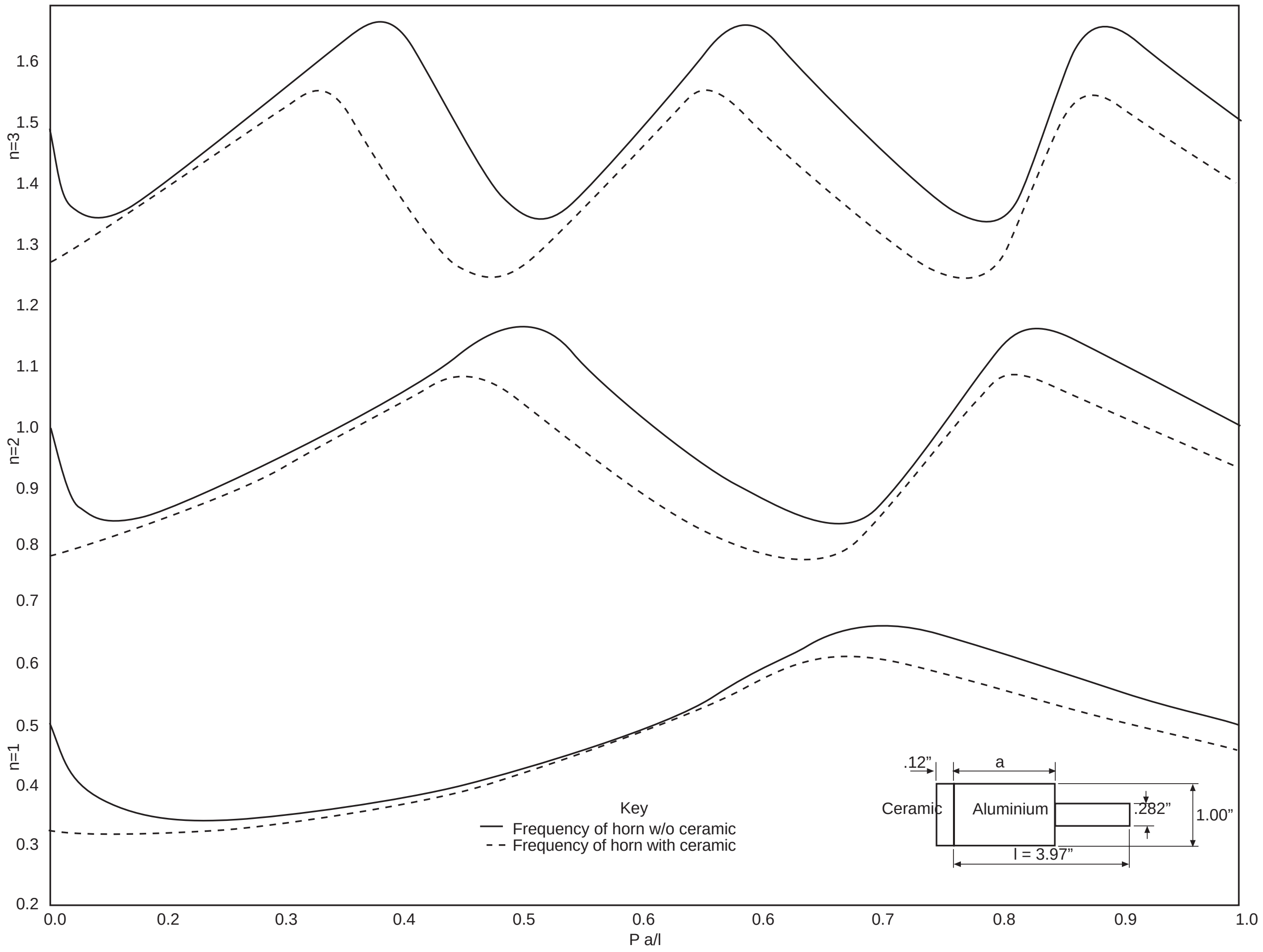


Fig. 3 First three resonant frequencies



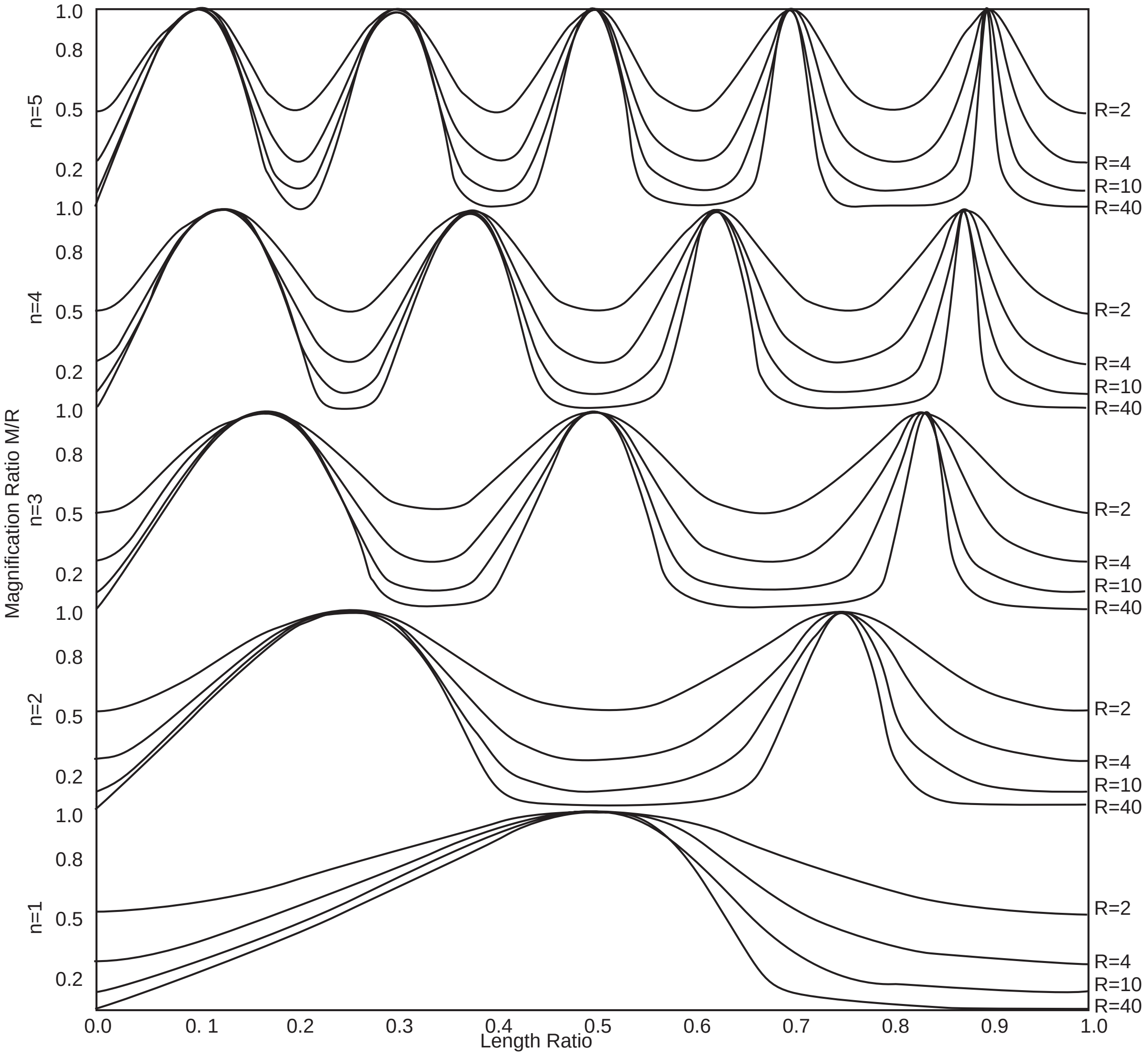
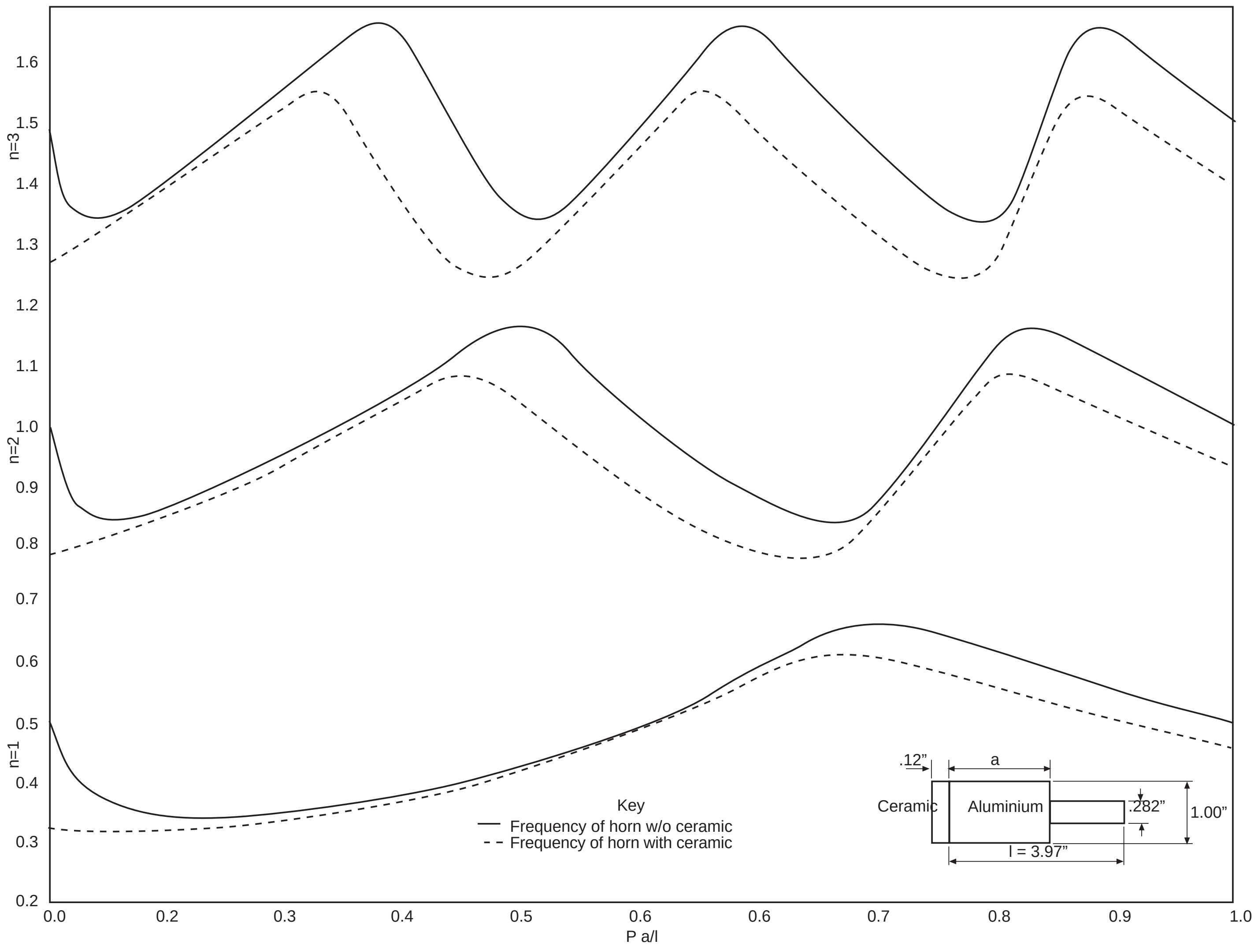


Fig. 4 Magnification of Stepped Horns


n=3
n=2
n=1
1.6
1.5
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.0
0.2
0.3
0.4
0.5
0.6
0.6
0.7
0.8
0.9
1.0
P a/l
Key
Frequency of horn w/o ceramic
Frequency of horn with ceramic
.12”
a
Ceramic
Aluminium
.282”
1.00”
l = 3.97”

THE STEPPED HORN

ERRATA

p. 814 At the end of the second line of Section III, "tion" is omitted from the word "magnification".

p. 815 At the beginning of the last paragraph of Section III, the parentheses should read "by a factor of 3:1 for the fundamental in the limiting case of $R = \infty$".

p. 822 Fig. 7 $v = y/\rho$ should read $v^2 = y/\rho$

ADDENDUM

The computed data for frequency and magnification were received after the publication deadline for this paper. Since that time, new graphs have been made and the attached figs. 3, 4, and 5 should replace figs. 3 and 4 of the text.

In fig. 3 it is interesting to note, for the limiting case of $R = \infty$, that the frequency maxima of the n th overtone coincide with the frequency minima of the (n+l) th overtone. In fact, for this limiting case, the frequency curves are given by the two sets' of hyperbolas

$$q=m/2p \qquad q=(2m-1)/4(1-p) \qquad m=1, 2, 3\ldots$$

These are shown as broken lines in fig. 3.

In fig. 4 it should be observed that the magnification peaks are not symmetrical, and that, in the case of overtones, successive peaks are not identical.

In connection with fig. 5, Section VI, page 815 should read as follows:

"In order to check the validity of the solutions here given, a series of stepped horns was made with the step in different positions. These were driven by piezoelectric drivers, and the resonant frequencies, magnifications, and Q' s measured. Because of the presence of the driving transducer, it was not expected that the experimental results would agree with the theoretical results. Equation (1) of the Appendix does, however, give the frequencies of a three-section bar. This equation can be rewritten as

$$(ß+\gamma)\sin[k_b b + k_c c + Y] + (ß-\gamma)\sin[k_b b - k_c c + Y] = 0$$
$$Y = \tan^{-1}[\alpha/ß \ \tan k^a a]$$

In the present case of a ceramic transducer driving an aluminum stepped horn of the same major diameter,
$S_a= S_b= S, \quad \rho_b= \rho_c= \rho, \quad v_b=v_c= V$

and the frequency equation reduces to

$$B \sin[2\pi q+ Y] + (ß-\gamma)\sin[2\pi q(2p-1) + Y] = 0$$ where now
$$Y = \tan^{-1}[\rho_a v_a/\rho v \ \ \tan 2\pi q.\ v/v_a\ .\ a/l]$$

$$B = (R+1)/(R-1), \quad R=s/s_c, \quad p=b/l$$

The solution of this equation, for the particular case at hand, is plotted in fig. 5 (broken line) together with the solution for the case a = o (i.e. an infinitely thin transducer). In order to relate this to previous figures, the dimension b has been changed to a and the abscissa labelled accordingly. The experimental points show generally good agreement. The disagreements can probably be attributed to material inhamogeneity and ceramic-aluminum bonding variations, as well as to experimental error."

For computational purposes, it is convenient to rewrite the. frequency and magnification equations, for the stepped horn alone, as follows:

$$B \sin 2\pi q + \sin 2\pi q(2p-1) = 0$$

$$M = [1 + (R^2-1)\sin^2 2\pi qp]^{1/2}$$

where $B = (R+1)/(R-1)$

In this form it is fairly easy to find solutions to the frequency equation as the intersections of the sinusoid $B \sin 2\pi q$ with the family of sinusoids $\sin 2\pi q(2p-1)$. It can then be seen that the maximum and minimum values of q for any given value of B occur when $B\sin 2\pi q = \pm 1$. The corresponding p values may be found by substituting this into the frequency equation. Thus

$$q\ \begin{matrix}[\text{Max} +]\\ [\text{Min} -\ ]\end{matrix} = 1/2\ [n\pm\ 1/\pi\ .\ \sin^{-1}(1/B)]$$

$$p = 1/2[1\pm(-1)^{n+k}\ .\ (2k-1)/4q]$$

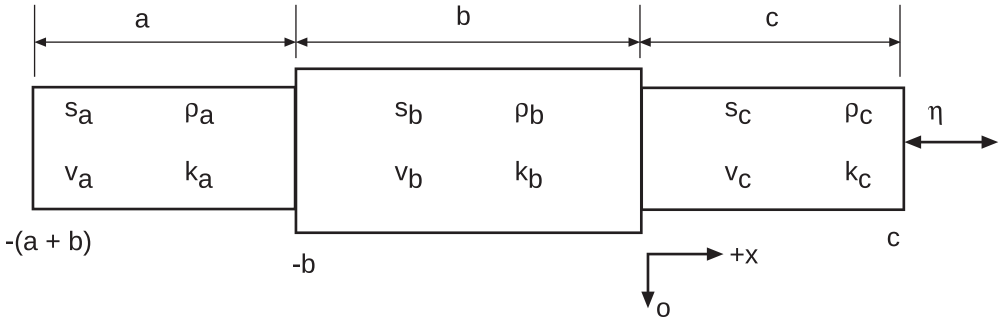
where n = order of overtone = 1, 2, 3…

k = 1, 2, 3 … n

$0\leq\sin^{-1} 1/B \leq \pi/2$

and the upper signs are used for maxima, the lower signs for minima.

From the p equation, the locus of the extrema may be plotted for reference on fig. 3.



S - AREA OF CROSS-SECTION
Y-ELASTIC MODULUS
ρ - DENSITY
v - LONGITUDINAL WAVE VELOCITY $v = Y/\rho = \omega^2/k^2$

FIG. 7 DERIVATION OF COMPOSITE BAR EQUATIONS